Evaluation of Definite Integrals

As seen in the last section, the evaluation of definite integrals by definition is quite complicated and time consuming. The following Theorem is an indispensable tool for the evaluation of definite integrals.

(5.4) Theorem. The Fundamental Theorem of Integral Calculus

If a function f is continuous on $[a, b]$ and there is a differentiable function F on $[a, b]$ such that $F'(x) = f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof. Let $P = \{x_0, x_1, \ldots x_{n-1}, x_n\}$ be a partition of $[a, b]$. Since $F'(x)$ exists on $[a, b]$, $F(x)$ is continuous on $[a, b]$. It follows that $F(x)$ is continuous on each subintervals $[x_{r-1}, x_r]$, $r = 1, 2, \ldots, n$ and $F'(x)$ exists therein. Hence, by the Mean Value Theorem, we have, for $r = 1, 2, \ldots, n$.

$$F(x_r) - F(x_{r-1}) = F'(c_r)(x_r - x_{r-1})$$

or

$$\sum_{r=1}^{n} [F(x_r) - F(x_{r-1})] = \sum_{r=1}^{n} F'(c_r)\, \Delta x_r.$$

The left-hand member of this equation is $F(b) - F(a)$. By hypothesis, $F'(x) = f(x)$ on $[a, b]$, so $F'(c_r) = f(c_r)$, $r = 1, 2, \ldots, n$. Therefore,

$$F(b) - F(a) = \sum_{r=1}^{n} f(c_r)\, \Delta x_r = S(P, f).$$

Taking limits as $n \to \infty$ and $\|P\| \to 0$, $S(P, f) \Rightarrow \int_a^b f(x)\, dx$, since f is continuous on $[a, b]$ and so $\int_a^b f(x)\, dx$ exists.

Thus, we have

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Now we take up the area problem.

Properties of Definite Integrals

Throughout this section f is a continuous function on $[a, b]$.

(5.6) Theorem. $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz.$

Proof. Let $F'(x) = f(x)$, $a \leq x \leq b$. Then by Theorem 5.4,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Also
$$\int_a^b f(z) = F(b) - F(a).$$

Hence the result.

(5.7) Theorem. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Proof. Let $F'(x) = f(x)$, $a \leq x \leq b$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad (1)$$

and
$$\int_b^a f(x)\,dx = F(a) - F(b). \qquad (2)$$

From (1) and (2), we have the required result.

(5.8) Theorem. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $c \in]a, b[$.

Proof. Let $F'(x) = f(x)$, $a \le x \le b$. Then

$$\int_a^c f(x)\,dx = F(c) - F(a) \text{ and } \int_c^b f(x)\,dx = F(b) - F(c).$$

Hence
$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = F(c) - F(a) + F(b) - F(c)$$

$$= F(b) - F(a) = \int_a^b f(x)\,dx.$$

(5.9) Theorem. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Proof. Let $x = a - z$. Then $dx = -dz$.

When $x = 0$, $z = a$ and when $x = a$, $z = 0$.

Therefore,
$$\int_0^a f(x)\,dx = -\int_a^0 f(a-z)\,dz = \int_0^a f(a-z)\,dz, \text{ by (5.7)}$$

$$= \int_0^a f(a-x)\,dx, \text{ by (5.6).}$$

(5.10) Theorem. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx.$

Proof.
$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx, \text{ by (5.8)}$$

In the second integral on the right of (1), put $x = 2a - z$. Then $dx =$-
When $x = a$, $z = a$ and when $x = 2a$, $z = 0$.

Therefore
$$\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-z)\,dz$$

$$= \int_0^a f(2a-x)\,dx, \text{ by (5.6) and (5.7).}$$

Substituting into (1), we have

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx.$$

(5.11) Theorem. (i) If $f(2a-x) = f(x)$, then $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$

(ii) If $f(2a-x) = -f(x)$, then $\int_0^{2a} f(x)\,dx = 0.$

Proof. (i) $$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx, \text{ by (5.10)}$$

$$= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx, \text{ by hypothesis}$$

$$= 2\int_0^a f(x)\,dx.$$

(ii) $$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx, \text{ by (5.10)}$$

$$= \int_0^a f(x)\,dx - \int_0^a f(x)\,dx, \text{ by hypothesis}$$

$$= 0$$

(5.12) Theorem. If $f(x) = f(a+x)$, then

$$\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$$

Proof. $$\int_0^{na} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx + \int_{2a}^{3a} f(x)\,dx + \cdots$$

$$+ \int_{(n-1)a}^{na} f(x)\,dx \qquad (1)$$